SPECTRAL UNIVERSES!
MATH POEM THOUGHT EXPERIMENT

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1. Continuous Realities

Let us suppose that the entire universe is continuous. Nothing changes suddenly, there are no boundaries. On close enough inspection, the transition from fish to ocean is gradual, the surface of our skin an illusion. The universe is just a collection of continuous functions $f_i(x, y, z)$ assigning attributes to every point in space. Maybe f_3 is temperature, f_7 is pressure, f_1 is brightness. Or they are more fundamental, or fewer. Maybe there is just one function, and its endless patterns present themselves to us as color, mass, heat. But whatever functions describe the state of the universe, let us suppose that they vary continuously in all directions, changing in very small spaces perhaps, but never jumping, each quantity ebbing and flowing through space.

And let us not only suppose that our reality is smooth in space, but that it *changes* only smoothly, that at each moment, *time* is just the quantities of the universe slipping imperceptibly, endless volumes flowing, stretching, sinking. Each characteristic we choose to consider is a boundless iceberg, melting. And if we focus on a single point, each of its attributes is changing continuously, its own small view of the vast flow.

Thus if we think of reality in spacetime, our f_i now four-dimensional functions $f_i(x, y, z, t)$, we have a continuous 'stack' of universes, each continuous in its spatial cross-section and continuous along time into the earlier and later universes, all of spacetime changing smoothly!

What a beautiful object! And each possible spacetime—each possible smooth progression of smooth universes—gives us another one! All the cross-sections of such an object, which are all the states the universe ever goes through in that reality, could include a vast variety of universes, going from emptiness to planets to a universe of cats floating through space to hot emptiness to cold fulness and perhaps anything we could imagine... This leads to my main question:

Question. *Could one timeline ever pass through* every *possible universe? That is, could one reality include every possible moment?*

Of course, we pose this question in the context of our restrictions—is there any *continuous* timeline through every *continuous* universe? Is there even enough 'room' on a timeline for every continuous universe? And if so is there a way to lay them out continuously?

2. Spectra

Think of a *spectrum*, a smoothly changing line of colors. Unlike the colors of a rainbow, this line goes on forever in both directions. Each color is just a frequency

FIGURE 1. Part of a spectrum, which would actually go on forever in both directions. Also, it would actually have no thickness, but is shown with a nonzero thickness here in the interest of visibility.

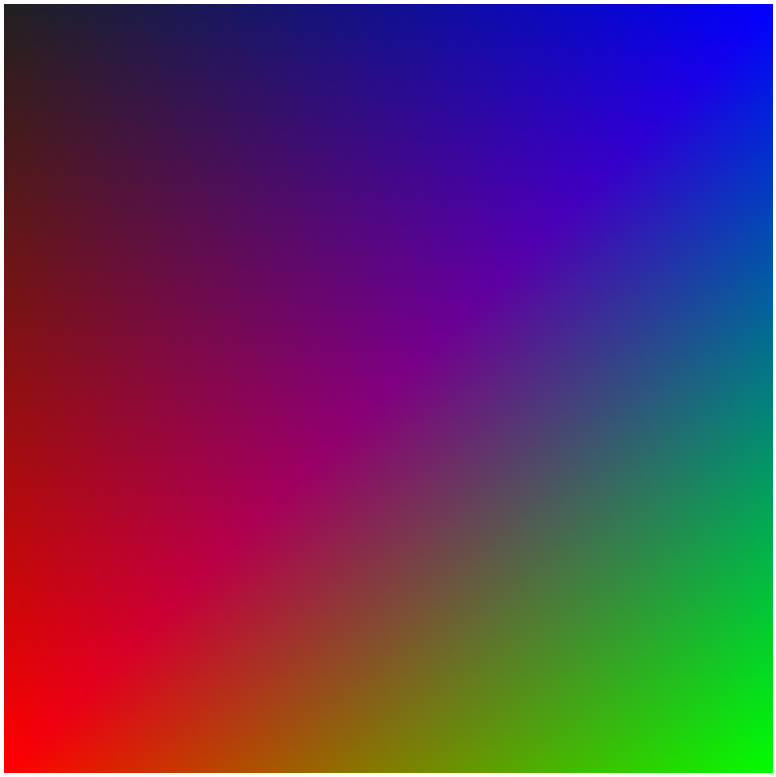

FIGURE 2. Part of a meta-spectrum, or spectrum of spectra, which would actually fill the plane. Each horizontal cross-section is a spectrum. The spectrum of Figure 1 is just one spectrum along this meta-spectrum, near the middle of the picture.

of light, so we can think of the frequency of the spectrum at position x as the value of a function $f(x)$. And thus saying that the spectrum is smooth is equivalent to saying that f is continuous. Part of a spectrum is shown in Figure 1.

Given a new concept, one should immediately consider its 'meta'. A meta-spectrum, or spectrum of spectra, would be a smoothly changing line of spectra. For example, the spectrum of Figure 1 might be just one spectrum along the meta-spectrum of Figure 2. As the figure or some thought makes clear, a meta-spectrum is equivalent to a smoothly colored plane.[1]

Note that we could alternatively interpret the spectrum 'colors' as heights. Then we realize that 'meta-spectra' are just continuous surfaces, the cross-sections of

[1] Since a meta-spectrum is just a smoothly colored plane, its orientation is not necessarily important; that is, a cross-section in any direction will yield a spectrum. We consider only *oriented* meta-spectra, in which the horizontal cross-sections are the spectra.

which, 'spectra', are just continuous functions. Thus we have two formulations of an interesting question about spectra:

Question. *Is there a meta-spectrum which includes every spectrum? Equivalently, is there a continuous surface which includes every continuous function as a cross-section?*

We will consider two aspects of this question: Is there even enough 'room' for every continuous function to fit in a continuous surface? And if so is there a way to arrange them continuously?

We prove that the answer to the first question is that yes, there is a way of numbering every continuous function with a unique real number, and thus they can be laid out on a line.

The second question, however, leads to a negative result: even though they could fit, there is no way to continuously pass through every continuous function. There is *no* meta-spectrum containing every spectrum. This is a bit sad, but at least we no longer have to hope and worry that it just *might* be true.

3. Cardinality

Question. *What is the cardinality of C^0, the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$?*

Lemma. *For $f, g \in C^0$, if f and g are equal on a dense subset $S \subseteq \mathbb{R}$, then $f = g$.*

In other words, a continuous function is completely described by its values on a dense subset.

Proof. Suppose f and g are equal on S. By continuity, for any $r \in \mathbb{R}$, $f(r) = \lim_{s\to r} f(s)$ where $s \in S$, and similarly $g(r) = \lim_{s\to r} g(s)$. But we always have $f(s) = g(s)$, so the limits are equal, and thus $f(r) = g(r)$.

So f and g are equal on all of $\mathbb{R}$, and thus they are the same element of C^0. □

This lemma allows us to easily answer our question.

Theorem 1. $C^0 \approx \mathbb{R}$

Proof. We prove both directions:

($C^0 \gtrsim \mathbb{R}$) For each $r \in \mathbb{R}$, we have the constant function $f_r \in C^0$ defined as $f_r(x) \equiv r$. Then clearly $\phi\colon \mathbb{R} \to C^0$ defined by $\phi(r) \equiv f_r$ is injective, so $C^0 \gtrsim \mathbb{R}$.

($C^0 \lesssim \mathbb{R}$) We consider the canonical mapping $\phi\colon C^0 \to \mathbb{R}^{\mathbb{Q}}$ (where $\mathbb{R}^{\mathbb{Q}}$ denotes the set of functions from $\mathbb{Q}$ to $\mathbb{R}$) defined by $\phi(f) \equiv (f \restriction \mathbb{Q})$. Then since $\mathbb{Q} \subset \mathbb{R}$ is dense, our lemma says that $\phi(f) = \phi(g) \iff f = g$, so ϕ is injective.

Now, $|\mathbb{R}^{\mathbb{Q}}| = |\mathbb{R}|^{|\mathbb{Q}|} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0} = |\mathbb{R}|$, so $C^0 \lesssim \mathbb{R}^{\mathbb{Q}} \approx \mathbb{R}$.

□

4. Continuity

Theorem 2. *There is no continuous surface which includes every continuous function as a cross-section.*

Proof. Communicated to the author by Barry Bradlyn.

Consider any continuous surface $F(x, y)$.

Let $f_y(x) = F(x, y)$ denote the horizontal cross-section of F at y.

Then consider the *diagonal* cross-section $g(x) = F(x, x)$. Since F is continuous, g is continuous. And note that for all y, $f_y(y) = g(y)$.

Let $h(x) = 1 + g(x)$. Since g is continuous, h is continuous. And for all y, $h(y) = 1 + g(y) = 1 + f_y(y) \neq f_y(y)$, so $h \neq f_y$.

Thus h is a continuous function which is unequal to every cross-section f_y, so we have found a continuous function which is not a cross-section of F.

So in general no continuous surface includes every continuous function as a cross-section. $\square$